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*Highlights (for review)

- The coefficient of variation appropriately measures inequality in voting settings.
- The coefficient of variation is appropriate to specify the inverse power problem.
- This specification is equivalent to using a particular distance-based error term.

Two-Tier Voting: Measuring Inequality and Specifying the Inverse Power Problem☆

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Abstract

There are many situations in which different groups make collective decisions by committee voting, with each group represented by a single person. This paper is about two closely related problems. The first is that of how to measure the inequality of a voting system in such a setting. The second is the inverse power problem: the problem of finding voting systems that approximate equal indirect voting power as well as possible. I argue that the coefficient of variation is appropriate to measure the inequality of a voting system and to specify the inverse problem. I then show how specifying the inverse problem with the coefficient of variation compares to using existing objective functions.

Keywords: measuring inequality, inverse power problem, indirect voting power, assembly of representatives

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1. Introduction

The term two-tier voting refers to situations where different groups have to make a collective decision and do so by voting in an assembly of representatives with one representative per group. Many decisions are taken daily through such voting by all kinds of institutions. The best-studied case is perhaps the Council of the European Union,[1] but it is by far not the only institution making use of some sort of two-tier voting. Other institutions include the UN General Assembly, WTO, OPEC, African Union, German Bundesrat, ECB, and thousands of boards of directors and professional and non-professional associations. The importance of two-tier voting is likely to further increase in the future. Globalization and the emergence of democracy in many parts of the world make collaboration in supra-national organizations more necessary and easier. Furthermore, modern communication technologies facilitate the organization in interest-groups, clubs, and associations, even when the members are geographically dispersed.

The question of how such two-tier voting systems should be designed remains unsolved and certainly cannot be solved in full generality. Nevertheless, there are theoretical concepts that provide guidelines, often stating which voting systems are fair. However, actual voting systems are never completely fair. It is then important to be able to measure how (un)equal a voting system is, i.e. how (un)equal the distribution of influence (or another variable of interest) is that a voting system generates. The inequality measure can then be used to compare voting systems within or across different populations. Such a measure could for example be used to investi-

[1]The literature on two-tier voting within the EU includes, among many others, Baldwin and Widgrén (2004), Beisbart et al. (2005), Felsenthal and Machover (2004), Laruelle and Valenciano (2002), Le Breton et al. (2012), Napel and Widgrén (2006), and Sutter (2000). For an overview of promising (voting) power research avenues see Kurz et al. (2015).

gate to what extent the inequality of voting systems correlates with other variables, such as income or crime rates. Furthermore, in some cases a voting system that is less equal than another one may have some advantages over the more equal one; for example it could be easier to explain its rules to citizens or this voting system could be more easily accepted by the people governed by it. It can then be important to be able to quantify by how much one voting system is more unequal than another one. I suggest to use the coefficient of variation to measure inequality in such voting settings. It can be applied to different variables of interest, such as indirect voting power (as measured by different power measures), the probability of a citizen's preferences to coincide with the voting outcome, or the number of representatives per citizen in an apportionment context.

Usually, no voting system exists that perfectly implements one of the abstract normative rules on the design of voting systems. The problem of finding voting systems that approximate these theoretical rules is called the inverse (power) problem. To specify the inverse problem, a measure is needed stating how well a voting system corresponds to a theoretical rule. I propose to use the coefficient of variation for this.[2] It turns out that minimizing the coefficient of variation leads to the same outcomes as minimizing the Euclidean distance (of normalized indirect voting power) from the fair ideal. This can be seen as support for the results achieved when using this distance (which cannot be used as an inequality measure in general, i.e. to compare inequality across different populations). This also

[2]I do not intend to develop algorithms solving the inverse power problem computationally given such a measure, which is what most of the literature does. Finding concrete solutions to the inverse power problem is not trivial; see for example Alon and Edelman (2010), De et al. (2012), Fatima et al. (2008), Kurz (2012), Kurz and Napel (2014), Leech (2003), and De Nijs and Wilmer (2012).

means that the coefficient of variation has a straightforward interpretation in the context of two-tier voting: It is a transformation of the Euclidean distance to the egalitarian ideal. I furthermore show that using an objective function based on (weighted) voting power at the group level to set up the inverse problem is unsatisfactory. For the discussion of the inverse problem I use a setting where equal indirect Banzhaf power is desired. However, the coefficient of variation can also be applied in a wide variety of other settings (the adaption to other settings is straightforward).

This paper is organized as follows. In Section 2, I describe one of the possible rules prescribing how voting systems should be designed (Penrose's Square Root Rule), which can then be used in the remainder for illustrations. In Section 3, I discuss what properties an inequality measure for voting systems should satisfy and why the coefficient of variation is an appropriate choice. In Section 4, I describe how the inverse power problem can be specified and discuss how this can be done based on the coefficient of variation. In Section 5, I illustrate the use of the coefficient of variation with examples and compare it to using different objective functions. Section 6 concludes.

2. One Theoretical Concept: Penrose's Square Root Rule

In this section, I introduce one theoretical, abstract rule on how voting systems should be designed, called Penrose's Square Root Rule. I will use this rule as an example in the next sections.[3]

[3] I use the most prominent rule on how two-tier voting systems should be designed, but using this rule as illustration does not mean that I endorse it as a normative concept. There are different possible criticisms of this rule, see for example Laruelle and Valenciano (2008). Furthermore, it has been shown that people do not necessarily like voting systems that accord with this rule (Weber, 2015).

There are N different groups, numbered from 1 to N, each group i consists of n_i individuals, numbered from 1 to n_i. Voting is binary, i.e. a proposal can either be accepted or rejected. Each individual favors the adoption of a proposal with probability one half, independently of all other individuals. Majority voting takes place within each group and the outcome determines the vote of the representative. The representatives of all groups come together in an assembly and it is determined according to their votes in combination with the voting system in the assembly of representatives whether the proposal is adopted or rejected.

Penrose's Square Root Rule: *The voting power of (the representative of) a group as measured by the Banzhaf index should be proportional to the square root of its population size.*

The main idea of this rule is to make it equally likely for each individual to influence the overall outcome of the two-tier voting procedure, independently of the group she belongs to. If a winning coalition turns into a losing coalition when voter j is excluded we say that voter j has a swing. The absolute Banzhaf index of a voter j is defined as the number of possible winning coalitions that turn into losing coalitions without voter j, divided by the total number of possible coalitions.[4] The normalized or relative Banzhaf index is the absolute Banzhaf index normalized so that the sum of the indices of all voters equals one.

Denote by Ψ_i^B the absolute Banzhaf power index of an individual in group i arising from majority voting in this group and by Φ_i^B the absolute Banzhaf power index of group i in the assembly of representatives, which depends on the voting system in place. Then the probability that an indi-

[4]In the scenario described here, the absolute Banzhaf index of a voter is the probability that this voter has a swing.

vidual in group i has a swing with respect to the overall outcome of the voting procedure (i.e. that she influences with her vote within the group the overall outcome) is Ψ_i^B times Φ_i^B, which is called the indirect Banzhaf voting power. Thus the probability of influencing the overall outcome is equal for all individuals if $\Psi_i^B \Phi_i^B$ is equal for all individuals or equivalently if

$$\Psi_i^B \Phi_i^B = \alpha \tag{1}$$

for some constant $\alpha > 0$ and all i.[5] It can easily be shown that equation (1) holds for all i if the normalized Banzhaf index of each group i is equal to

$$\frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^{N} \frac{1}{\Psi_j^B}}.$$

The normative rule on how to design voting systems as described here states that the indirect voting power $\Psi_i^B \Phi_i^B$ should be equal for all individuals independently of which group they are in, i.e. that equation (1) should hold for all i.[6]

3. Measuring the Inequality of Voting Systems

Voting systems in assemblies of representatives are in general not completely fair. Sometimes one may want to quantify how unequal a voting system is. Thus, an inequality measure for a voting system $\mathcal{W}$ in a population consisting of N groups with in total $m = \sum_{i=1}^{N} n_i$ individuals is needed.

[5]It is assumed that the grand coalition, i.e. all representatives voting together, can always pass a proposal. This excludes the trivial case $\alpha = 0$.

[6]The reason why this is usually referred to as square root rule is the following. Ψ_i^B in equation (1) can be approximated by $\sqrt{\frac{2}{\pi n_i}}$, thus equation (1) holds if the Banzhaf indices of the groups are proportional to the square root of population size.

I assume that there is a variable of influence or representation at the individual level $r = (r_1, ..., r_m)$ with all $r_i \geq 0$ and at least one r_i strictly positive. This variable could for example be indirect Banzhaf power as described in Section 2 so that $r = \Psi^B \Phi^B = (\Psi_1^B \Phi_1^B, ..., \Psi_m^B \Phi_m^B)$. This variable could also be something different, such as for example indirect Shapley-Shubik power or the probability of being successful rather than influential (see e.g. Laruelle and Valenciano, 2008). It could also be the number of representatives per citizen in an apportionment context as for example for the US House of Representatives (see Balinski and Young, 2001). Then one can define the measure of inequality of a two-tier voting system (with a very slight abuse of notation) as $\lambda(\mathcal{W}, n_1, ..., n_N) := \lambda(r)$.

Such an inequality measure $\lambda(r)$ should satisfy certain axioms (for general treatments of inequality measures see e.g. Atkinson, 1970, or Cowell, 2011). Important axioms are:

Anonymity: $\lambda(r_1, ..., r_m) = \lambda(r_{k_1}, ..., r_{k_m})$ for any permutation $(k_1, ..., k_m)$ of (1,...,m). This axiom states that all individuals are equally important for the inequality measure.

Scale Invariance: $\lambda(r) = \lambda(\gamma r)$ for any $\gamma > 0$. This axiom states that the unit of measurement of influence (or representation) should not matter for the inequality measure.

Population Principle: $\lambda(r_1, ..., r_m) = \lambda(\overbrace{r_1, ..., r_1}^{k}, \overbrace{r_2, ...r_2}^{k}, ..., \overbrace{r_m, ..., r_m}^{k})$. This axiom states that if a population is an identical multiplication of another one with respect to the influence each individual has, both populations (with their voting systems) should be judged to be equally unequal.

Principle of Transfers: $\lambda(r_1, ..., r_{k_i}, ..., r_{k_j}, ..., r_m) > \lambda(r_1, ..., r_{k_i} + h, ..., r_{k_j} - h, ..., r_m)$ for any $h > 0$ and $i, j \in \{1, ..., m\}$ with $r_{k_i} + h \leq r_{k_j} - h$. This axiom states that the inequality measure should decrease if one can decrease the influence of one citizen by a bit while simultaneously increasing the influence of another citizen who has less influence by the same amount (assuming that the redistribution does not change the ordering of influence between these two citizens).

There are multiple well-known inequality measures that satisfy these axioms (e.g. the Gini index, the coefficient of variation, or the Theil index). I will focus here on the coefficient of variation and argue that it is a good choice to measure inequality of voting systems.

The coefficient of variation is defined as the ratio of the (population) standard deviation σ to the (population) mean μ, $cv = \frac{\sigma}{\mu}$, thus in our case

$$cv(r) = \frac{\sqrt{\frac{1}{m}\sum_{i=1}^{m}(r_i - \bar{r})^2}}{\bar{r}}.$$

It is thus the inverse of the signal-to-noise ratio. The coefficient of variation satisfies all of the axioms stated above.

A further property of the coefficient of variation is that redistributing influence at any end of the distribution reduces (or increases) the inequality measure by the same amount.[7] This can be seen as an advantage of the coefficient of variation over other inequality measures such as the Gini index.[8] Furthermore, some researchers and certainly most politicians concerned with voting systems are not specialists in inequality measurement. It

[7]More precisely, an infinitesimal transfer from an individual with influence y_1 to an individual with influence $y_1 - h$ will always have the same effect on the coefficient of variation independently of where y_1 lies in the distribution. For the Gini index, for example, this effect depends on the distribution with usually larger effects in the middle of the distribution than in the tails (see Atkinson, 1970).

[8]An advantage of the Gini index contributing to its popularity in measuring income or wealth inequality is the fact that it can handle negative values (such as debt). However, this advantage of the Gini index plays no role when measuring the inequality of a voting system, because influence and representation are generally non-negative.

is therefore important to have a salient and easily understandable measure at hand; this also suggests that the coefficient of variation is to be preferred over other measures satisfying above axioms. Anticipating a finding of the next section, the coefficient of variation also has a straightforward interpretation in two-tier voting settings as a transformation of the Euclidean distance of normalized indirect voting power from the fair ideal.

4. The Inverse Power Problem in Two-Tier Voting Settings

If one wants to find a voting system which optimally approximates equal indirect voting power, the inverse power problem needs to be solved. This problem is specified with an error term (or objective function) describing how much a voting system deviates from equal indirect voting power.[9] I first describe possible error terms and then discuss specifying the inverse problem with the coefficient of variation. In this section, I assume that indirect Banzhaf power is the variable of interest, but using the coefficient of variation is by no means restricted to such a setting (the relation to apportionment is made in Footnote 13).

4.1. Error Terms Based on Voting Power on the Group Level

The system of equations (1) usually does not hold exactly for any voting system. It is thus necessary to find a voting system approximating full equality. One way to do this is to take a voting system that minimizes the deviation of the normalized Banzhaf index of each group from the vector that would yield equal indirect voting power. One can then take the Eu-

[9]It is of course possible to address the inverse problem only within a subset of voting systems. Such a subset could for example be all weighted voting systems, all weighted voting systems satisfying some additional conditions (e.g., no player can be a dummy player or the quota can be at most two thirds), or all double majority voting systems.

clidean distance as error term (i.e. as objective function) or equivalently its square, which leads to the minimization of

$$err_{group,basic}\left(\Psi^B,\Phi^B\right) := \sum_{i=1}^{N}\left(\frac{\Phi_i^B}{\sum_{j=1}^{N}\Phi_j^B} - \frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^{N}\frac{1}{\Psi_j^B}}\right)^2. \tag{2}$$

Such a squared error term at the group level has been frequently used in the literature.[10]

It is easily seen that this cannot be the best term to minimize. The groups have different sizes and the idea is to equalize voting power at the individual level. I will now propose a first way to fix this (as I will show later on, the easy fix does not work perfectly). This easy fix consists of weighing the squares in the error term by their group size. In order for this error term not to increase with the number of groups or the group sizes, one can divide by the total number of individuals. One can furthermore take the square root, so that the error term is measured in the 'unit' of indirect voting power rather than in its square (taking the square root only matters if one is interested in quantities, not if one is solely interested in ranking voting systems). This leads to the minimization of

$$err_{group,imp}\left(\Psi^B,\Phi^B\right) := \sqrt{\frac{1}{\sum_{i=1}^{N} n_i}\sum_{i=1}^{N} n_i\left(\frac{\Phi_i^B}{\sum_{j=1}^{N}\Phi_j^B} - \frac{\frac{1}{\Psi_i^B}}{\sum_{j=1}^{N}\frac{1}{\Psi_j^B}}\right)^2}. \tag{3}$$

[10] See for example Barthélémy and Martin (2011), Kirsch and Langner (2011), Leech (2002), Turnovec (2011), or Życzkowski and Słomczyński (2013). Note that the main scientific contributions of these works are not corrupted by using this suboptimal error term.

4.2. An Error Term Based on Normalized Indirect Voting Power

Another error term sometimes used in the literature (e.g. Le Breton et al., 2012, Maaser and Napel, 2007) is as follows. Rather than deriving the power distribution at the group level that leads to equal indirect voting power at the individual level, one considers indirect voting power $\Psi_i^B \Phi_i^B$ directly. One then normalizes this index of indirect voting power, so that it sums up to one when added up over all individuals. This yields a 'normalized indirect voting power index' of the form

$$\frac{\Psi_i^B \Phi_i^B}{\sum_{j=1}^N n_j \Psi_j^B \Phi_j^B}.$$

Then one chooses the voting system that minimizes the sum of the squared deviations of this index from one over the number of individuals, so that one ends up minimizing

$$err_{indirect}\left(\Psi^B, \Phi^B\right) := \sum_{i=1}^N n_i \left(\frac{\Psi_i^B \Phi_i^B}{\sum_{j=1}^N n_j \Psi_j^B \Phi_j^B} - \frac{1}{\sum_{j=1}^N n_j} \right)^2. \tag{4}$$

Again, one could take the square root (yielding the Euclidean distance of the normalized indirect power vector from $1/\sum_{j=1}^N n_j$), but it is usually left out.

4.3. Using the Coefficient of Variation to Specify the Inverse Problem

Starting out a bit differently, the following way to specify the inverse power problem seems natural. One is looking for a voting system where indirect voting power is as equal as possible. I propose to choose the voting system that directly minimizes the inequality of indirect voting power, thus $\lambda(\Psi^B \Phi^B)$ for an inequality measure λ. Potentially any inequality measure could be used here, such as for example the Gini index. However, as argued in Section 3, the coefficient of variation is an appropriate measure of inequality for voting systems, thus I propose to minimize $cv(\Psi^B\Phi^B)$.

Now, I briefly show how using the coefficient of variation can also be derived in a similar way to motivating the objective functions above. If the system of equations (1) holds, all individuals have equal (indirect) voting power. Keeping in mind that the error at the individual level is what we are interested in, one can then minimize

$$\sum_{i=1}^{N}\sum_{j=1}^{n_i}\left(\Psi_i^B\Phi_i^B-\alpha\right)^2=\sum_{i=1}^{N}n_i\left(\Psi_i^B\Phi_i^B-\alpha\right)^2. \tag{5}$$

over different voting systems. Equal indirect voting power corresponds to equation (1) holding for all i, no matter what the exact value of α is. Therefore it is natural to give each voting system its 'best shot', i.e. to let α depend on the voting system (so that both α and Φ^B depend on the voting system):

$$\alpha=\arg\min_{\gamma}\sum_{i=1}^{N}n_i\left(\Psi_i^B\Phi_i^B-\gamma\right)^2.$$

It can easily be shown that then

$$\alpha=\frac{1}{\sum_{i=1}^{N}n_i}\sum_{i=1}^{N}n_i\Psi_i^B\Phi_i^B=:\overline{\Psi^B\Phi^B}. \tag{6}$$

Note that $\overline{\Psi^B\Phi^B}$ is the mean of $\Psi^B\Phi^B$ (taken at the individual level). Minimizing expression (5) with α as in (6) can still be adjusted. To make the error term independent of the number of groups and the group sizes, one can divide by the number of individuals. Furthermore, as for $err_{group,imp}$, if one wants to measure the variation of indirect voting power in the same unit as indirect voting power rather than its square, one can take the square root. Finally, it is desirable that the scale used does not change the outcome, i.e.

that merely multiplying the indices Φ^B of all groups with a constant does not change the outcome. This can be achieved by dividing through $\overline{\Psi^B\Phi^B}$. This leads to an error term that is then equal to the coefficient of variation of indirect voting power:

$$cv\left(\Psi^B\Phi^B\right) := \frac{\sqrt{\frac{1}{\sum_{i=1}^N n_i}\sum_{i=1}^N n_i\left(\Psi_i^B\Phi_i^B - \overline{\Psi^B\Phi^B}\right)^2}}{\overline{\Psi^B\Phi^B}}. \tag{7}$$

It turns out that for any given population (for the same $N, n_1, \ldots, n_N$) cv and $err_{indirect}$ are just monotonic transformations of each other.[11] This means that when addressing the inverse problem, either of the two leads to the same results. This is not self-evident, if one were to choose a different inequality measure this result would in general not hold. However, as I have argued, the coefficient of variation is a good inequality measure for voting systems; this equivalence can thus be seen as support for the results from using $err_{indirect}$.[12] Note, however, that $err_{indirect}$ is not appropriate to measure inequality across different populations.[13]

[11]It is $cv(\Psi^B\Phi^B) = \sqrt{m \cdot err_{indirect}(\Psi^B, \Phi^B)}$, with $m = \sum_{i=1}^N n_i$.

[12]At the same time, if one considers using the Euclidean distance of normalized indirect voting power from the fair ideal a good way to specify the inverse problem, this equivalence can be seen as a motivation for using the coefficient of variation (and not, say, the Gini index) as inequality measure.

[13]The relation to an apportionment setting where representatives per citizen is the variable of interest is as follows. Using Webster's method is equivalent to minimizing the error term $\sum_{i=1}^N n_i\left(a_i/n_i - h/\sum_{j=1}^N n_j\right)^2$ as proposed by Sainte-Lagüe, where a_i is the number of seats for group/state i and h is the total number of seats to be apportioned (see Balinski and Young, 2001). This error term is the same as $err_{indirect}$ and minimizing it yields thus the same result as minimizing the coefficient of variation, which can be seen as support for Webster's method. However, using the error term proposed by Sainte-Lagüe does not give a good measure to compare the inequality of different apportionments across different populations or different house sizes for the same reasons $err_{indirect}$ does not constitute a good measure to compare inequality across populations for voting power, which will be illustrated in Sections 5.2 and 5.3.

5. Illustrations

In this section I will illustrate with three (hypothetical) examples that the coefficient of variation is suitable to specify the inverse problem and to measure the inequality of voting systems. The first example is concerned with the inverse problem, where cv and $err_{indirect}$ lead to the same outcome. I show how using the coefficient of variation (or equivalently $err_{indirect}$) is to be preferred over using $err_{group,imp}$ (as argued above, $err_{group,basic}$ is clearly not optimal, therefore I do not consider it here). In the second and third examples, I compare the inequality of two voting systems in different populations. Here, in contrast to using cv, using $err_{indirect}$ does not yield convincing results (similarly, the Euclidean distance would not yield convincing results).

5.1. First Example: Mean-Preserving Spread

There are six groups, numbered from 1 to 6. Groups 1 and 2 have ten members each, the other groups have five members. This means that in the first stage (the election of the representatives) individuals have voting power $\Psi^B_{1,2} = 0.2460938$ and $\Psi^B_{3,4,5,6} = 0.375$, respectively. Indirect voting power would be equal across all individuals if the voting systems were such that

$$\frac{\Phi^B_{1,2}}{\sum_{i=1}^6 \Phi^B_i} = 0.2162162 \qquad \text{and} \qquad \frac{\Phi^B_{3,4,5,6}}{\sum_{i=1}^6 \Phi^B_i} = 0.1418919.$$

Now we compare two (hypothetical) voting systems $\mathcal{W}_1$ and $\mathcal{W}_2$. The voting systems are such that the normalized Banzhaf indices are as follows:

$$\frac{\Phi_1^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} = 0.2162162 + 0.05, \qquad \frac{\Phi_2^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} = 0.2162162 - 0.05,$$
$$\frac{\Phi_{3,4,5,6}^B(\mathcal{W}_1)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_1)} = 0.1418919, \qquad \text{and}$$
$$\frac{\Phi_{1,2}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} = 0.2162162, \qquad \frac{\Phi_{3,4}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} = 0.1418919 + 0.05,$$
$$\frac{\Phi_{5,6}^B(\mathcal{W}_2)}{\sum_{i=1}^6 \Phi_i^B(\mathcal{W}_2)} = 0.1418919 - 0.05.$$

Assume for simplicity and to have a nice illustration that normalized and absolute Banzhaf indices are equal. Now we can calculate the indirect voting power of each individual, depending on the group she is in. This yields

$$\Psi_1^B \Phi_1^B(\mathcal{W}_1) = 0.06551414, \qquad \Psi_2^B \Phi_2^B(\mathcal{W}_1) = 0.04090477,$$
$$\Psi_{3,4,5,6}^B \Phi_{3,4,5,6}^B(\mathcal{W}_1) = 0.05320946, \qquad \text{and}$$
$$\Psi_{1,2}^B \Phi_{1,2}^B(\mathcal{W}_2) = 0.05320946, \qquad \Psi_{3,4}^B \Phi_{3,4}^B(\mathcal{W}_2) = 0.07195946,$$
$$\Psi_{5,6}^B \Phi_{5,6}^B(\mathcal{W}_2) = 0.03445946.$$

One can easily see that using $err_{group,imp}$ does not distinguish between the two voting systems, both would be judged to be 'equally equal' ($err_{group,imp}$ equals $\sqrt{1/120}$ for both voting systems). If one looks carefully at the indirect voting power, this does not seem justified, though. For both voting systems, there are twenty individuals with indirect voting power 0.05320946, which is also the mean of indirect voting power under both voting systems. For both voting systems, there are ten individuals with higher voting power and 10 with lower power. The absolute difference between the higher value and the middle value is always equal to the difference between the middle value and the lower value; however, these differences are higher under the second voting system than under the first. The first voting system is thus less unequal than the second one. It is also selected correctly by the coefficient of variation, $cv(\Psi^B\Phi^B(\mathcal{W}_1)) = 0.1635184$ and $cv(\Psi^B\Phi^B(\mathcal{W}_2)) = 0.249171$.

5.2. Second Example: Comparing Inequality across Different Populations I

The first population consists of six groups of four people each. The second population consists of four groups of eight people each. Assume that the voting system in place in the first population, $\mathcal{W}_X$, is such that indirect voting power is

$$\Psi^B_{1,2,3}\Phi^B_{1,2,3}(\mathcal{W}_X) = 0.03 \qquad \text{and} \qquad \Psi^B_{4,5,6}\Phi^B_{4,5,6}(\mathcal{W}_X) = 0.01.$$

In the second population, the voting system, $\mathcal{W}_Y$, is such that indirect voting power is

$$\Psi^B_{1,2}\Phi^B_{1,2}(\mathcal{W}_Y) = 0.03 \qquad \text{and} \qquad \Psi^B_{3,4}\Phi^B_{3,4}(\mathcal{W}_Y) = 0.01.$$

This means that in the first population half of the individuals have indirect voting power of 0.03, while the other individuals have voting power 0.01. The same holds for the second population. Which of the two voting systems is more unequal? The only reasonable answer seems to be that they are 'equally unequal' (as stated in the population principle in Section 3). Using the coefficient of variation as inequality measure also yields this result, $cv(\Psi^B, \Phi^B(\mathcal{W}_\mathcal{X})) = 0.5 = cv(\Psi^B, \Phi^B(\mathcal{W}_\mathcal{Y}))$. However, if one tried to use $err_{indirect}$ as a measure of inequality, one would obtain

$err_{indirect}(\Psi^B, \Phi^B(\mathcal{W}_\mathcal{X})) = 0.0069444$ and $err_{indirect}(\Psi^B, \Phi^B(\mathcal{W}_\mathcal{Y})) = 0.0078125$. One would then misleadingly conclude that the voting system in the first population is more equal than the one in the second population.

5.3. Third Example: Comparing Inequality across Different Populations II

After already illustrating in the last hypothetical example that the coefficient of variation is well-suited to compare the inequality of voting systems across different populations (while for example $err_{indirect}$ is not), I will make a similar point now with an example where also the voting systems in the assemblies are specified. In this example, all voting will be majority voting, not only at the group level but also in the assemblies.

In the first population there are two groups of five and two groups of three individuals. In the assembly of representatives any three representatives can pass a proposal. In the second population there are three groups of five and three groups of three individuals. Here, any four (of six) representatives can pass a proposal.

This means that in the first population the voting power at the group level is $\Psi^B_{1,2} = 0.375$ and $\Psi^B_{3,4} = 0.5$. In the second population, these values are $\Psi^B_{1,2,3} = 0.375$ and $\Psi^B_{4,5,6} = 0.5$. Majority voting in the assembly of representatives leads to $\Phi^B_{1,2,3,4}(\mathcal{W}_{maj-4}) = 0.375$ in the assembly of the first population and to $\Phi^B_{1,2,3,4,5,6}(\mathcal{W}_{maj-6}) = 0.3125$ in the assembly of the second population. Thus, we have

$$\Psi^B_{1,2}\Phi^B_{1,2}(\mathcal{W}_{maj-4}) = 0.140625 \qquad \text{and} \qquad \Psi^B_{3,4}\Phi^B_{3,4}(\mathcal{W}_{maj-4}) = 0.1875$$

in the first population. In the second population, we have

$$\Psi^B_{1,2,3}\Phi^B_{1,2,3}(\mathcal{W}_{maj-6}) = 0.1171875 \text{ and } \Psi^B_{4,5,6}\Phi^B_{4,5,6}(\mathcal{W}_{maj-6}) = 0.15625.$$

This means that in both populations, five eighth of the population have lower indirect voting power than the remaining three eighth. In both populations, the voting power of the part of the population with higher power is exactly a third higher than the voting power of the rest.[14] Thus, these two populations exhibit the same degree of inequality. Accordingly, the coefficient of variation is equal for both populations with these voting systems (it is equal to 0.1434438 in both cases). If one were to judge the inequality by $err_{indirect}$, one would conclude that the second population with its voting system is more equal (the values are 0.0012860 and 0.0008573, respectively).

6. Concluding Remarks

In this short paper I have first addressed the question of how the inequality of voting systems should be measured. I have argued that the coefficient of variation is an appropriate measure. Then I have argued that it is appropriate to specify the inverse power problem with the coefficient of variation when a fair voting system is desired. This is to be preferred over minimizing error terms that are based on weighted or unweighted voting power at the group level. It turns out that specifying the inverse problem with the coefficient of variation is equivalent to using an objective function based on the distance of the normalized indirect voting power from the fair

[14] Note that the relative comparisons are relevant and not the absolute ones. In absolute terms the difference in voting power is smaller in the second population. Voting power arguments are in general based upon relative comparisons, i.e. arguments are usually of the form that the indirect voting power in one group is, say, twice as large as the indirect voting power in another group. If the arguments were based on absolute values, discussions about voting power would lose their meaning when groups are large (e.g. in the case of the EU, where indirect voting power is minuscule). Also note that common inequality measures take a relative perspective and that the principle of scale invariance would be violated if an absolute perspective were taken.

ideal. Unlike the coefficient of variation, such an objective function cannot be used, however, to compare the inequality of voting systems across different populations. I have used a setting where equal indirect Banzhaf voting power is desired as illustration, but the coefficient of variation can be applied in many different settings.

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